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On a system of Seiberg–Witten equations

Fortuné Massamba[a,b], George Thompson[a,*]

[a] *The Abdus Salam ICTP, High Energy Group, Strada Costiera 11, P.O. Box 586, 34100 Trieste, Italy*
[b] *I.M.S.P., B.P. 613, Porto-Novo, Benin*

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Abstract

We introduce and study a system of Seiberg–Witten equations. These are r copies of the usual Seiberg–Witten equations coupled to each other involving r connections on r $\text{Spin}_\mathbb{C}$ structures as well as r positive spinors and are Abelian generalizations of the Seiberg–Witten equations. For $r = 2$, we show that the moduli space of solutions is a compact, orientable and smooth manifold. For minimal surfaces of general type, we are able to determine the basic classes.

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1. Introduction

The Seiberg–Witten (SW) equations are made out of sections of a $\text{Spin}_\mathbb{C}$ structure and a connection on a line bundle [17]. The rather difficult theory of Donaldson requires a vector bundle of rank 2. Even more complicated are the non-Abelian monopole theories of Pidstragach (see [7]), which is believed to bridge the gap between the SW and Donald-

* Corresponding author. Tel.: +39 040 2240381; fax: +39 040 2240304.
E-mail address: surname@ictp.it (G. Thompson).

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son invariants, and of Labastida and Mariño [15] which generalize both. A survey of the invariants can be found in [7]. The aim of this paper is to introduce a rank 2 theory which is nevertheless Abelian.

The equations that we introduce can be considered in a slightly broader context. Let E be rank r vector bundle on a compact closed four-manifold X. Fix a Riemannian structure on X, (g, X) and denote the self-dual 2-forms by $\Omega^2_+(X)$. Fix also a $\mathrm{Spin}_{\mathbb{C}}$ structure on X. Let ψ be a section of $W^+ = \mathrm{Spin}_{\mathbb{C}}(X) \otimes E$. For ϕ and λ in W^+, let $q : W^+ \times W^+ \to \Omega^2_+(X) \otimes \mathrm{End}\, E$ be the trace free part of the endomorphism $\theta \mapsto \langle\theta, \phi\rangle\, \lambda$. Let Φ be a section of End (End E). The equations of interest are

$$\begin{aligned} &F_A^+ + \Phi \cdot q(\psi, \psi) = 0,\\ &\not{D}_{\Gamma+A}\psi = 0,\\ &d_A\Phi = 0 \end{aligned} \tag{1.1}$$

where A is a connection on E, Γ the Levi–Cevita connection on $\mathrm{Spin}_{\mathbb{C}}$ and d_A is the covariant derivative on End (End E) with the connection induced from that on E.

One possible solution to these equations would have Φ equal to a scalar times the identity endomorphism. In this case, on a Kähler manifold, the equations become (up to a perturbation) equivalent to a set of equations discussed in [4]. Those equations are shown to have a notion of stability.

If Φ is not proportional to the identity endomorphism then, to have a solution to the last equation in (1.1), the bundles must split. The equations that we consider in this paper correspond to such a situation. The equations thus obtained are described in detail in the next section but we briefly summarise them here. In this case, we have $\mathrm{Spin}_{\mathbb{C}}(X) \otimes E = \oplus_i(L_1^{E_{i1}} \otimes \cdots \otimes L_r^{E_{ir}} \otimes S^+) = \oplus_i(\mathbf{L}_i \otimes S^+)$, and neither the spin bundle S^+ nor the line bundles L_i need exist. However, $L_i^{\otimes 2}$ are honest line bundles and the E_{ij} are integers so that the combinations $\mathbf{L}_i \otimes S^+$ are bundles. Let M_i be sections of the bundles $S^+ \otimes \mathbf{L}_i$, $2A_i$ be connections on the line bundles $L_i^{\otimes 2}$ and $2\mathbf{A}_i$ be connections on $\mathbf{L}_i^{\otimes 2}$. The system of r Seiberg–Witten equations are

$$\begin{aligned} &F_{A_i}^+ + \textstyle\sum_j D^{ij}\, q(M_j, M_j) = 0\\ &\not{D}(\mathbf{A}_i)\, M_i = 0, \end{aligned} \tag{1.2}$$

with D_{ij} a non-singular and not necessarily integral matrix. The choice of Φ in (1.1) fixes the matrix D.

The value of r is called the rank of the system of equations. We will often refer to the system of r Seiberg–Witten equations as the rank r SW equations.

The equations under consideration were proposed in the context of studying the Rozansky–Witten invariants on a three-manifold, Y [2]. Higher rank equations of this type should correspond to higher rank Rozansky–Witten invariants, that is to higher order LMO or Casson invariants [8]. One would expect that considering these equations on $X = Y \times S^1$ one would get something like the Euler characteristic of a suitable Floer theory. This was part of our motivation for studying the higher rank case on a four-manifold.

Here is a brief summary of the contents of the text. Along the way, we highlight where new ingredients, beyond those required for the rank 1 SW, are used. In Section 2, the equations are introduced. There is also a discussion on the form of Φ that we consider as well as a comment on what happens to the equations under a conformal change of the metric. In Section 3, the virtual dimension of the moduli space is computed with the help of the index theorem and basic classes are defined. The question of reducible points in the moduli space is also addressed there. All of this is standard.

The compactness of the space of solutions is established in Section 4 following the approach of Witten [17]. However, the discussion here to obtain a priori bounds on the curvature 2-forms and on the sections (and their derivatives) is somewhat more involved. In the following section, the perturbed equations are introduced and we mostly follow Chapter 6 of [12] to establish that the parameterized moduli space, for $b_2^+(X) > 1$, is compact and smooth and that it is essentially independent of the metric and of generic perturbations.

With the general picture in hand we make a small side excursion in Section 6 to show, by way of examples, why we have made such a particular choice for the form of Φ. In this section, we make use of the fact that, by specializing, one can have a theory of rank s and with N sections. For example, for $s < N$, take $r = N$ in the equations and set $r - s$ line bundles L_a to be trivial with connections A_a taken to be zero and also set $D_{ai} = 0$ with $a = s + 1, \dots, r$.

In Section 7, we specialize to Kähler manifolds. One can mimic to some extent the work done on the rank 1 equations. There is a moment map description of the moduli space, however, we have not been able to establish that the bundles are 'stable' in some appropriate sense. Instead, one uses a trick to establish that given a holomorphic section on the Kähler manifold $(\omega,\ X)$ one obtains a solution to the equations on $(e^{2\rho}\omega,\ X)$ for some conformal factor ρ see Proposition 7.8. This is a rather weak result but it nevertheless allows us to prove that the basic classes of the rank 2 SW equations on a minimal surface of general type are a subset of the Cartesian product of the allowed rank 1 SW classes, i.e. subsets of the four classes $(\pm K_X, \pm K_X)$, see Proposition 7.11.

Here is a brief summary of what is not included in the text. We do not analyze the situation for the equations on other types of manifolds. For example, neither general symplectic manifolds nor hermitian non-Kähler manifolds are considered. The techniques introduced by Taubes [16] and by Biquard [1] presumably apply in the present setting as well. We do not define rank r SW invariants, though they can be defined in the natural way, as we do not use them.

Bounds on the sections and curvatures, though not presented here, can also be obtained when the rank is greater than 2. This can be found in the thesis [11].

A rather serious deficiency is that there are no applications to topology.

In the text, we will sometimes refer to a Fierz identity. That is an identity on the tensor product of the Clifford algebra and it reads

$$\begin{aligned} 4\mathbb{I}_{\alpha\beta}\,\mathbb{I}_{\rho\sigma} &= \mathbb{I}_{\alpha\sigma}\,\mathbb{I}_{\rho\beta} + \sum_{\mu}((\gamma_\mu)_{\alpha\sigma}(\gamma^\mu)_{\rho\beta} - (\gamma_\mu\gamma_5)_{\alpha\sigma}(\gamma^\mu\gamma_5)_{\rho\beta}) + (\gamma_5)_{\alpha\sigma}(\gamma_5)_{\rho\beta} \\ &\quad - \sum_{\mu\nu}\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\sigma}(\sigma^{\mu\nu})_{\rho\beta} \end{aligned} \tag{1.3}$$

1.1. Note added

After the completion of this manuscript, two references were brought to our attention. In [9] quiver theories which correspond to special cases of the rank r SW equations with $E_{ij} = 0, \pm 1$ have been studied. The conditions for stability of vortex type equations on a Kähler surface have been established in [5]. The system of equations we study should be an example of those in [5] but we have not been able to show this. However, if true, then one would have stability in hand and one could forgo the analysis of Section 7 and certainly strengthen the results there.

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2. The equations

We fix an oriented, compact, Riemannian four-manifold X. We start with r, possibly non-existent, line bundles L_i, $i = 1, \dots, r$, on X, so that $S^+ \otimes L_i$ are $\mathrm{Spin}_{\mathbb{C}}$ structures on X for all i. However, the $\mathrm{Spin}_{\mathbb{C}}$ structures of interest are

$$\mathbf{L}_i \otimes S^+ = L_1^{E_{i1}} \otimes \cdots \otimes L_r^{E_{ir}} \otimes S^+.$$

The matrix E_{ij} may well be the identity matrix, though in general we only demand that $\det E \neq 0$, that the entries be integers and they are such that the $L_i^{\otimes 2}$ are honest line bundles. Summing over all tuples $(L_1, \dots, L_r)$ for a general matrix E means that one does not sum over all possible tuples of $\mathrm{Spin}_{\mathbb{C}}$ structures on X. However, for $E \in SL(r, \mathbb{Z})$ then one does sum over all such tuples of $\mathrm{Spin}_{\mathbb{C}}$ structures.

The model we have of the map $TX \to \mathrm{Hom}(S^+, S^-)$ (which is well defined even if $S^\pm$ are not) is

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_3 + \sqrt{-1}\,x_4 & x_1 - \sqrt{-1}\,x_2 \\ x_1 + \sqrt{-1}\,x_2 & -x_3 + \sqrt{-1}\,x_4 \end{pmatrix},$$

and this fixes our conventions for the Dirac matrices.

Let $2A_i$ be connections on the line bundles $L_i^{\otimes 2}$, with an abuse of language we will say that the A_i are connections on L_i. The connection forms are $\sqrt{-1}A_i$ so that the A_i are real. Denote by M_i charged positive chirality spinors, that is sections of the bundles $S^+ \otimes \mathbf{L}_i$.

The rank r Seiberg–Witten equations are

$$F^+_{A_i} + \sum_j D^{ij}\, q(M_j, M_j) = 0 \tag{2.1}$$

$$\not{D}(\mathbf{A}_i)\, M_i = 0. \tag{2.2}$$

where, in local coordinates,

$$q_{\mu\nu}(M_i, M_i) = \frac{\sqrt{-1}}{2}\left(\bar{M}_i \sigma_{\mu\nu} M_i\right)$$

and $\mathbf{A}_i = \sum_j E_{ij} A_j$ is a connection on $\mathbf{L}_i$. The map q is the same one as in the introduction but written in local coordinates.

Some comments are in order.

Remark 2.1. The two matrices E and D that appear in the equations are related. That relation is dictated by wishing to emulate the use of the Weitzenbock trick to get a vanishing theorem as in the case of the rank 1 equations. The condition on the matrices is that $D^{-1} \cdot E$ be a symmetric positive definite matrix (See Section 4). Infact, the matrix D need not have integer entries.

Remark 2.2. Though not strictly necessary we impose the further condition that D^{-1} have integral entries. With this assumption in hand we can write (2.1) as

$$F^+_{\mathbf{B}} = -q(\mathbf{M}, \mathbf{M}) \tag{2.3}$$

with $\mathbf{B} = D^{-1} \cdot A$, and so that B_i is a connection on $L_1^{\otimes D^{-1}_{i1}} \otimes \cdots \otimes L_r^{\otimes D^{-1}_{ir}}$.

Note that conformal classes of a metric on X yield related equations. Denote the Dirac operator and sections on (g, X) by $\not{D}$ and M_i (as above) and those on $(e^\rho g, X)$ by $\not{D}^\rho$ and M_i^ρ. The rank r SW equations on $(e^\rho g, X)$ are

$$F^+_{\mathbf{B}} = -q(\mathbf{M}^\rho, \mathbf{M}^\rho,), \qquad \not{D}(\mathbf{A}_i)^\rho M_i^\rho = 0.$$

Note that the Hodge star operator acting on 2-forms is conformal invariant and so the $+$ superscript is the same for (g, X) and $(e^{2\rho} g, X)$.

Proposition 2.3. *Let the rank r SW equations for the Riemannian manifold* (g, X) *be as above. The equations for* $(e^{2\rho} g, X)$ *are*

$$F^+_{\mathbf{B}} = -e^{-\rho} q(\mathbf{M}, \mathbf{M}), \quad \not{D}(\mathbf{A}_i) M_i = 0,$$

with $M_i^\rho = e^{-3\rho/2}\, M_i$.

Proof. The relationship between M_i^ρ and M_i follows from the scaling dimension of the spinors. That the first equation holds is obvious. That the second holds follows from the fact that under a conformal scaling the Dirac operator behaves as

$$\not{D}^\rho = e^{-5\rho/2}\not{D}e^{3\rho/2}. \quad \square$$

Our aim is to study the space of solutions of (a perturbed version of) the rank 2 equations. We will sum over all L_i, so that we do not need to specify which $\mathrm{Spin}_{\mathbb{C}}$ structures we are dealing with at the outset.

3. The moduli space and the basic classes

Let $\mathcal{G}_i$ denote the gauge group of bundle automorphisms of L_i. The space of gauge transformations, $\mathcal{G}$, is the product of these spaces of bundle automorphisms,

$$\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_r.$$

Each of the $\mathcal{G}_i$ is a copy of $\mathrm{Map}(X, U(1))$ and their complexifications are copies of $\mathrm{Map}(X, \mathbb{C}^*)$. The space of solutions to the rank r SW equations is left invariant under $\mathcal{G}$.

By moduli space, we mean the space of solutions to the rank r SW equations modulo gauge transformations. Let (A_i, M_i) be a solution to the rank r equations. We want to use an index calculation to determine the dimension, d, of the moduli space at that point. For this we need only linearize the equations about the solution. The linearized equations, however, are simply r copies of linearized rank 1 equations, with bundles $\mathbf{L}_i$.

The operator that arises on linearizing the equation for the self-dual curvature and the gauge fixing condition is

$$T_0 = d + d^* : \Omega^1(X, \mathbb{R}) \to \Omega^0(X, \mathbb{R}) \oplus \Omega^2_+(X, \mathbb{R}). \tag{3.1}$$

The linearization of the Dirac equation for a section of $S^+ \otimes L$, on dropping terms of order zero, is

$$T_1(L) = \not{D}(A) : \Gamma(S^+ \otimes L) \to \Gamma(S^- \otimes L). \tag{3.2}$$

The index of T_0 is $d_0 = -(\chi + \tau)/2$ and that of $T_1(L)$ is $d_1(L) = -\tau/4 + c_1(L)^2$. The virtual dimension of a given moduli space of rank r with line bundles L_i and n sections M_a is

$$\begin{aligned} d(L_1, \ldots, L_r) &= r\, d_0 + \sum_{a=1}^{n} d_1(\mathbf{L}_a) \\ &= -\frac{2r\chi + (2r + n)\sigma}{4} + \sum_{i,j} C^{ij} c_1(L_i) c_1(L_j), \end{aligned} \tag{3.3}$$

with $C_{ij} = \sum_a E_{ai} \cdot E_{aj}$, that is $C = E^{\mathrm{T}} \cdot E$.

The usual rank 1 SW moduli space with one section of charge one has virtual dimension

$$d(L) = -\frac{2\chi + 3\tau}{4} + c_1(L)^2.$$

We have

Proposition 3.1. *The virtual dimension of the rank 2 equations is*

$$\begin{aligned} d(L_1, L_2) &= 2d_0 + d_1(\mathbf{L}_1) + d_1(\mathbf{L}_2) \\ &= -\frac{2\chi + 3\tau}{2} + (E^{\mathrm{T}} \cdot E)^{ij} c_1(L_i)\, c_1(L_j). \end{aligned} \tag{3.4}$$

Definition 3.2. The basic classes are

$$x = (x_1, \ldots, x_r) = (-c_1(\mathbf{L}_1^{\otimes 2}), \ldots, -c_1(\mathbf{L}_r^{\otimes 2})).$$

Note that in the special case that $E_{ij} = \delta_{ij}$ the equations decouple and the moduli spaces have virtual dimensions $d_0 + d_i$ separately for each i. In that case, the basic invariants are essentially r-tuples of the usual SW basic classes.

The equations have a number of symmetries. Apart from the gauge symmetry which was discussed above there is also invariance under

$$M_i \to \overline{M}_i, \qquad A_i \to -A_i \tag{3.5}$$

for all i simultaneously. The transformation on the connections really corresponds to exchanging the line bundles L_i with L_i^{-1}. Consequently, we have

Proposition 3.3. *If* $(x_1, \ldots, x_r)$ *is a basic class then so too is* $(-x_1, \ldots, -x_r)$.

A solution to the rank r SW equations would be reducible, if one or more of the sections is zero. Reducibility arises since then constant gauge transformations do not act. Suppose that one of the sections is zero, say M_1. Then, (2.3) reads

$$F^+(B_1) = 0$$

that is B_1 is an Abelian instanton. If $b_2^+(X) \geq 1$ then, generically, the intersection of $\mathrm{H}^2(X, \mathbb{Z})$ with $\mathrm{H}^2_-(X, \mathbb{R})$ is the zero class. This means that the connection B_1 is flat. This possibility will not arise with the introduction of a perturbation to the equations as given in Section 5.

4. Weitzenbock formulae, a priori bounds and compactness

In this section, we will obtain bounds on $F^i_\pm$ and on $|M_i|$ which will allow us to conclude that the moduli space of solutions is compact. Then, we give our prescription for the orientation of the moduli space.

We begin with a squaring argument. Set

$$s_{i\mu\nu} = F^{i+}_{\mu\nu} + \tfrac{i}{2}\textstyle\sum_{j=1}^{2} D^{ij}\,(\bar{M}_j\sigma_{\mu\nu}M_j)$$
$$k_i = \not{D}(E_{ij}A_j)\,M_i,$$

and, for a solution to the SW equations we must have

$$\int_X \mathrm{d}^4x\sqrt{g}\sum_{i=1}^{2}\left(\frac{1}{2}G^{ij}\,s_j\cdot s_i + |k_i|^2\right) = 0, \tag{4.1}$$

with $G = E^{\mathrm{T}}\cdot D^{-1}$ a symmetric and positive definite matrix.

Using the fact that,

$$\not{D}(E_{ij}A^j)^2\,M_i = D^\mu\,D_\mu\,M_i + \frac{i}{2}\sum_{j=1}^{2}\,E_{ij}\,F^j_{\mu\nu}\cdot\sigma^{\mu\nu}\,M_i - \frac{1}{4}RM_i, \tag{4.2}$$

we find that (4.1) becomes

$$\int_X \mathrm{d}^4x\sqrt{g}\sum_{i=1}^{2}\left(\frac{1}{2}G^{ij}s_j\cdot s_i + |k_i|^2\right)$$
$$= \int_X \mathrm{d}^4x\sqrt{g}\sum_{i,j=1}^{2}\left(\frac{1}{2}G^{ij}F^{i+}F^{j+} - \frac{1}{8}\sum_{\mu,\,\nu}\bar{M}_i\sigma_{\mu\nu}M_i\,B_{ij}\bar{M}_j\sigma^{\mu\nu}M_j + \delta_{ij}|D\,M_i|^2\right.$$
$$\left.+\frac{1}{4}\,R\,\delta_{ij}|M_i|^2\frac{1}{8}\sum_{\mu,\,\nu}\right) = 0, \tag{4.3}$$

with $B = D^{\mathrm{T}}\cdot E^{\mathrm{T}} = D^{\mathrm{T}}\cdot G\cdot D$ a positive definite symmetric matrix. At this point, the choice of the matrix G becomes evident. It was chosen, so that the 'mixed term' $F_+\cdot\bar{M}\sigma M\cdot E$ would drop out of the equations.

A small calculation using a Fierz identity for the gamma matrices (1.3) allows to write

$$-\frac{1}{8}\sum_{j,k=1}^{2}\sum_{\mu,\,\nu}\bar{M}_j\sigma_{\mu\nu}M_j\,B_{jk}\bar{M}_k\sigma^{\mu\nu}M_k = \frac{1}{2}\sum_{j,k=1}^{2}(2|\bar{M}_jM_k|^2 - |M_j|^2|M_k|^2)\,B_{jk}. \tag{4.4}$$

It is easy to check that

$$(2|\bar{M}_1M_2|^2 - |M_1|^2|M_2|^2)B_{12} \le |B_{12}|\,|M_1|^2|M_2|^2,$$

so that

$$-\frac{1}{8}\sum_{j,k=1}^{2}\sum_{\mu,\,\nu}\bar{M}_j\sigma_{\mu\nu}M_j\,B_{jk}\bar{M}_k\sigma^{\mu\nu}M_k \ge \frac{1}{2}\left(\sqrt{B_{11}}|M_1|^2 - \sqrt{B_{22}}|M_2|^2\right)^2$$
$$+\left(\sqrt{B_{11}B_{22}} - |B_{12}|\right)|M_1|^2|M_2|^2$$

where $\sqrt{B_{ij}}$ always represents the positive root. We can cast (4.1) in the form of an inequality,

$$\int_X \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}\sum_{i,j=1}^{2} G_{ij}F^{i+}F^{j+} + \sum_{i=1}^{2}|D\,M_i|^2 + \frac{1}{2}\left(\sqrt{B_{11}}|M_1|^2 - \sqrt{B_{22}}|M_2|^2\right)^2 + \left(\sqrt{B_{11}B_{22}} - |B_{12}|\right)|M_1|^2|M_2|^2 + \frac{1}{4}R\sum_{i=1}^{2}|M_i|^2\right) \leq 0. \tag{4.5}$$

One immediate consequence of (4.5) is that, just as for the usual SW invariants, there are no solutions apart from the trivial ones if R is non-negative.

We use this equation to prove that the moduli space of interest is compact. In order to do so it useful to re-write it once more. Add $\int \mathrm{d}^4x\sqrt{g}R^2/32\lambda$ to both sides of (4.5) with λ a constant. Then, we have

$$\int_X \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}\sum_{i,j=1}^{2} G_{ij}F^{i+}F^{j+} + \sum_{i=1}^{2}|D\,M_i|^2 + \frac{1}{2}\left(\sqrt{B_{11}^{\lambda}}|M_1|^2 - \sqrt{B_{22}^{\lambda}}|M_2|^2\right)^2 + \left(\sqrt{B_{11}^{\lambda}B_{22}^{\lambda}} - |B_{12}| - \lambda\right)|M_1|^2|M_2|^2 + \frac{1}{2\lambda}\left(\lambda(|M_1|^2 + |M_2|^2) + \frac{R}{4}\right)^2\right) \leq \frac{1}{32\lambda}\int \mathrm{d}^4x\sqrt{g}R^2 \tag{4.6}$$

where $B_{ii}^{\lambda} = B_{ii} - \lambda$ (no sum). This equation can be checked easily by expanding everything out. The advantage in expressing the inequality in this way is that if one takes

$$0 < \lambda \leq \frac{\det B}{2|B_{12}| + B_{11} + B_{22}}$$

then every term in the integrand is positive semi-definite. The only thing we need to check is that

$$\sqrt{B_{11}^{\lambda}B_{22}^{\lambda}} \geq |B_{12}| + \lambda.$$

Squaring this expression, we are led to the restriction on λ above.

Consequently, each term in the integrand is separately bounded by $\int \mathrm{d}^4x\sqrt{g}R^2/32\lambda$. We can now see that the sections M_i have bounded norm since

$$\int_X \mathrm{d}^4x\sqrt{g}\,\frac{1}{2}(B_{11}^{\lambda}|M_1|^4 + B_{22}^{\lambda}|M_2|^4) = \int_X \mathrm{d}^4x\sqrt{g}\,\frac{1}{2}\left(\sqrt{B_{11}^{\lambda}}|M_1|^2 - \sqrt{B_{22}^{\lambda}}|M_2|^2\right)^2 + \int_X \mathrm{d}^4x\sqrt{g}\sqrt{B_{11}^{\lambda}B_{22}^{\lambda}}|M_1|^2|M_2|^2\left(1 + \frac{\sqrt{B_{11}^{\lambda}B_{22}^{\lambda}}}{\sqrt{B_{11}^{\lambda}B_{22}^{\lambda}} - |B_{12}| - \lambda}\right) \frac{1}{32\lambda}\int \mathrm{d}^4x\sqrt{g}R^2, \tag{4.7}$$

(with λ less than its allowed maximal value). Hence, the norms of the sections and their derivatives are bounded.

To complete the discussion, we want to show that the basic classes are also bounded. We have, for each L_i, the bound

$$\int_X \mathrm{d}^4x\sqrt{g}\,\frac{1}{2}\sum_{i,j=1}^{2} G_{ij}F^{i+}F^{j+} \le \frac{2|B_{12}|+B_{11}+B_{22}}{32\det B}\int \mathrm{d}^4x\sqrt{g}R^2,$$

while the dimension formula gives us a bound on $|F^{i-}|$. To obtain this bound, we first use the Cauchy–Schwarz inequality to deduce that

$$C^{12}\int_X \mathrm{d}^4x\sqrt{g}F^1_-F^2_- \ge -|C^{12}|\,\|F^{1-}\|\,\|F^{2-}\|,$$

and

$$C^{12}\int_X \mathrm{d}^4x\sqrt{g}F^1_+F^2_+ \le |C^{12}|\,\|F^{1+}\|\,\|F^{2+}\|$$

where for any form ω its norm is

$$\|\omega\| = \left(\int_X \mathrm{d}^4x|\omega|^2\right)^{1/2}.$$

The dimension formula reads

$$C^{ij}\,c_1(L_i)c_1(L_j)[X] \ge \frac{1}{2}(2\chi+3\tau),$$

that is,

$$\begin{aligned}C^{11}\|F^{1-}\|^2 + C^{22}\|F^{2-}\|^2 - 2|C^{12}|\,\|F^{1-}\|\,\|F^{2-}\| &\le C^{11}\|F^{1+}\|^2 + C^{22}\|F^{2+}\|^2\\ &\quad +2|C^{12}|\,\|F^{1+}\|\,\|F^{2+}\| - 2\pi^2\,(2\chi+3\tau)\,.\end{aligned} \tag{4.8}$$

Denote the right-hand side of the inequality (4.8) by S (it is bounded by our previous results), and note that we can express the inequality as

$$(\sqrt{C_{11}}\|F^{1-}\| - \sqrt{C_{22}}\|F^{2-}\|)^2 + 2(\sqrt{C_{11}C_{22}} - |C_{12}|)\|F^{1-}\|\,\|F^{2-}\| \le S \tag{4.9}$$

The left-hand side is a sum of positive terms, so in particular we have $S \ge 0$ and

$$(\sqrt{C_{11}}\|F^{1-}\| - \sqrt{C_{22}}\|F^{2-}\|)^2 \le S, \qquad 2(\sqrt{C_{11}C_{22}} - |C_{12}|)\|F^{1-}\|\,\|F^{2-}\| \le S.$$

It is now straightforward to deduce that

$$\|F^{1-}\|^2 \le C_{11}^{-1}\cdot H\cdot S, \qquad \|F^{2-}\|^2 \le C_{22}^{-1}\cdot H\cdot S, \tag{4.10}$$

with

$$H = 1 + \frac{\sqrt{C_{11}C_{22}}}{\sqrt{C_{11}C_{22}} - |C_{12}|}.$$

We have therefore the following:

Proposition 4.1. *The L^2 norms of M_i, $D_{\mathbf{A}_i}M_i$, $F^-_{\mathbf{A}_i}$ and $F^+_{\mathbf{A}_i}$ are bounded.*

Since X is compact we also get pointwise norms on the sections and the curvatures. One can now follow the discussion in Chapter 5.3 of [12] to establish that the moduli space is compact.

Proposition 4.2. *The moduli space of rank 2 Seiberg–Witten equations is compact.*

The moduli space of solutions can be oriented in the same way as for the rank 1 moduli space [17]. An orientation at a point in the solution space is the same thing as the trivialization of the determinant of the linearization operator; direct sums of T_0 and $T_1(\mathbf{L}_i)$. We do not need to trivialize the determinant line of $T_1(\mathbf{L}_i)$ as each of those is naturally trivial as explained by Witten. To trivialize the determinant of T_0 one fixes on an orientation of $\mathrm{H}^1(X, \mathbb{R}) \oplus \mathrm{H}^2_+(X, \mathbb{R})$. Having picked such an orientation we have then trivialized $(\det T_0)^{\otimes r}$.

Proposition 4.3. *The moduli space of rank 2 Seiberg–Witten equations is orientable.*

5. Perturbed equations

The moduli space of solutions to the rank 2 SW equations may not be smooth. Furthermore, the expected dimension of the moduli space may not be the actual dimension. To get around these problems, one perturbes the equations. We perturb as for the rank 1 equations, namely the first SW equation, (2.3) becomes

$$F^+(\mathbf{B}) = -q(\mathbf{M}, \mathbf{M}) + \mathbf{h} \tag{5.1}$$

with $\mathbf{h} = (h^1, h^2)^{\mathrm{T}}$ two generic real C^∞ self-dual 2-forms on X.

We denote the moduli space of perturbed solutions, modulo the action of the gauge group $\mathcal{G}$, by $\mathcal{M}(\mathbf{L}, \mathbf{h})$.

Proposition 5.1. *For a fixed metric and a generic perturbation, the perturbed equations do not allow for reducible solutions if $b_2^+(X) > 0$.*

Proof. A reducible solution requires one of the sections to be zero. Without loss of generality let $M_1 = 0$. We have that $F^+(\mathbf{B}_1) = h_1$. However, the harmonic part of $F(\mathbf{B}_1)/2\pi$ is an integral class so if the harmonic part of h_1 does not lie on the integral lattice then there are no solutions. □

One can now mimic the discussion on the parametrized moduli space of Chapter 6 in [12]. We summarize that discussion (references in this paragraph are to [12]). Fix a $\mathrm{Spin}_{\mathbb{C}}$ structure. For the rank 1 SW equations (with $E = D = 1$), one introduces a map $F : \mathcal{A} \times \Omega^2_+(X) \to \Omega^2_+(X) \oplus S^- \otimes L$ given by

$$F(A, M, h) = (F_A^+ + q(M, M) - h, \not{D}_A M)$$

where $\mathcal{A}$ is the space of connections on L Cartesian product with the space of sections of $S^+ \otimes L$. Suitable Sobolev norms being given on $\mathcal{A} \times \Omega^2_+(X)$. For the section $M \neq 0$, one shows that the differential of the map DF is onto, Lemma 6.2.1. Proposition 6.2.2 then establishes that the parameterized (by h) moduli space consisting of all irreducible pairs of $([A, M], h)$ for which the perturbed SW equations are satisfied is a smooth manifold. This manifold is a fibre bundle over the parameter space $\Omega^2_+(X, \mathbb{R})$ with fibre $\mathcal{M}^*(L, h)$ the moduli space of irreducible solutions to the rank 1 SW equations modulo gauge equivalence for fixed perturbation. The differential of the projection mapping is Fredholm and its index is

$$4\, d(L) = c_1(L^{\otimes 2})^2 - 2\chi(X) - 3\tau(X).$$

These results follow immediately from Lemma 6.2.1. The role of Corollary 6.2.3 is to establish that the fibre for a generic perturbation is smooth. This too is straightforward to establish, with the main ingredient being an application of the Sard–Smale theorem.

We only need, therefore, to generalize Lemma 6.2.1 of [12] to the rank 2 case. The proof of the following proposition follows closely that given for the rank 1 equations in [12] and so is not given in detail. Let $\mathcal{A}$ denote the space of connections on $L_1 \otimes L_2$ Cartesian product with the space of sections of $S^+ \otimes (\mathbf{L}_1 \oplus \mathbf{L}_2)$.

Proposition 5.2. *Let $F : \mathcal{A} \times \Omega^2_+(X) \times \Omega^2_+(X) \to \Omega^2_+(X) \oplus \Omega^2_+(X) \oplus S^-(\mathbf{L}_1 \oplus \mathbf{L}_2)$ be given by*

$$F(\mathbf{A}, \mathbf{M}, \mathbf{h}) = (F_{\mathbf{B}}^+ + q(\mathbf{M}, \mathbf{M}) - \mathbf{h}, \not{D}_{\mathbf{A}}\mathbf{M}). \tag{5.2}$$

At any point $(\mathbf{A}, \mathbf{M}, \mathbf{h})$ for which $F(\mathbf{A}, \mathbf{M}, \mathbf{h}) = 0$ and $\mathbf{M} = (M_1 \neq 0, M_2 \neq 0)$ the differential of the map DF is onto.

Proof. Let $(\mathbf{a}, \mathbf{m}, \mathbf{k})$ be tangent vectors, then

$$DF(\mathbf{a}, \mathbf{m}, \mathbf{k}) = (d^+\mathbf{b} + q(\mathbf{m}, \mathbf{M}) + q(\mathbf{M}, \mathbf{m}) + \mathbf{k}, \not{D}_{\mathbf{A}}\mathbf{m} + \not{\mathbf{a}}\mathbf{M}),$$

with $\mathbf{b} = D^{-1} \cdot E^{-1}\mathbf{a}$. This is onto on the first factor, as can be seen by varying $\mathbf{k}$. So, our task is to keep the first factor fixed and to show that then DF is onto on the second factor. Since the Dirac operator is invertible outside the zero mode set we have that DF is onto in the second factor except possibly for modes that satisfy the Dirac equation, that is, those in kernel of the Dirac operator $\not{D}_{\mathbf{A}}$ on $S^- \otimes \mathbf{L}$. Let $N_i \in S^- \otimes \mathbf{L}_i$ be in the kernel of the Dirac operator. Suppose, furthermore, that the N_i are L^2 orthogonal to the image of the map

$$G : (\mathbf{a}, \mathbf{m}) \mapsto \not{D}_{\mathbf{A}}\mathbf{m} + \not{\mathbf{a}}\mathbf{M}.$$

We take the N_i to be non-zero (if N_i, for some i, is zero then it is in the image of G). Since the Dirac operator is elliptic this means that the N_i do not vanish on any open subset. Pick a small enough open ball U so that the $\mathbf{N}$ and $\mathbf{M}$ are non-zero there. We have a map $(S^+ \otimes L) \otimes (S^- \otimes L^{-1}) \to \Omega^1(X, \mathbb{C})$ given by Clifford multiplication,

$$(M, N) \mapsto N \cdot \gamma_\mu \cdot M\, dx^\mu.$$

Consider the vectors $\mathbf{v}$ given by

$$v^i_\mu = N_i \gamma_\mu M_i,$$

and set $\mathbf{a} = \mathrm{Re}\,\mathbf{v}$. Note that both $\mathrm{Im}\,\mathbf{v}$ and $\mathrm{Re}\,\mathbf{v}$ are non-zero on U. Consequently,

$$\mathrm{Re} < N_i, \not{a}_i M_i >= \mathrm{Re} \int_X N_i \not{a}_i M_i = \int_X |a_i|^2 > 0.$$

But this means that the N_i are not L^2 orthogonal to $G(a_i, 0)$. This is a contradiction and so the orthogonal compliment to the image of DF is trivial and hence the N_i are in the image of DF and the map is onto. □

It remains to establish that the moduli space $\mathcal{M}(\mathbf{L}, \mathbf{h})$ for any $\mathbf{h}$ is compact. A small variation on the arguments used in Section 4 give us the required,

Proposition 5.3. *For solutions to the perturbed rank 2 SW equations the L^2 norms of M_i, $D_{\mathbf{A}_i} M_i$, $F^-_{\mathbf{A}_i}$ and $F^+_{\mathbf{A}_i}$ are bounded.*

Proof. For the perturbed Eq. (4.1) becomes

$$\int_X \mathrm{d}^4x \sqrt{g} \sum_{i=1}^{2} \left(\frac{1}{2} G^{ij} \overline{s}_j \cdot s_i + |k_i|^2 \right) = \int_X \mathrm{d}^4x \sqrt{g} \sum_{i=1}^{2} \frac{1}{2} G^{ij}\, h_j \cdot h_i.$$

Consequently, following the steps after (4.1), we are led to the same equations as before except that one should make the replacement

$$\frac{1}{32\lambda} \int_X \sqrt{g}\, R^2 \to \frac{1}{32\lambda} \int_X \sqrt{g}\, R^2 + \int_X \mathrm{d}^4x \sqrt{g} \sum_{i=1}^{2} \frac{1}{2} G^{ij}\, h_j \cdot h_i,$$

and the bounds obtained are those of Section 4 with this substitution understood. □

Putting all the pieces together, we have:

Proposition 5.4. *The moduli space $\mathcal{M}(\mathbf{L},\mathbf{h})$ of solutions to the rank 2 SW equations on X with $b_2^+(X)>0$ for a generic value of* $\mathbf{h}$ (*avoiding the reducible connections*) *is a smooth compact manifold.*

One can also show along the lines of the proof of Theorem 6.5.1 of [12]:

Proposition 5.5. *Let X be a closed compact smooth four-manifold with $b_2^+(X)>1$. Let g_t be a smooth path of metrics connecting g_0 and g_1 and let $\mathbf{h}_t$ be a smooth and generic path of self-dual 2-forms connecting $\mathbf{h}_0$ and $\mathbf{h}_1$. Suppose that for $(g_0,\mathbf{h}_0)$ and for $(g_1,\mathbf{h}_1)$ Proposition 5.2 holds. The parametrized moduli space $\mathcal{M}(L_1,L_2,\mathbf{h}_t)$ of solutions to the parameterized equations*

$$F_{\mathbf{B}}^{+_t}=q(\mathbf{M},\mathbf{M})+\mathbf{h}_t,\qquad \not{D}_{\mathbf{A}}^t\mathbf{M}=0,$$

where $+_t$ means the Hodge star operator for the metric g_t and $\not{D}_{\mathbf{A}}^t$ means the Levi–Cevita part of the connection is also that associated to g_t. Then, $\mathcal{M}(L_1,L_2,\mathbf{h}_t)$ consists only of irreducible points and is a smooth compact manifold whose boundary is the disjoint union of the moduli spaces associated to $(g_0,\mathbf{h}_0)$ and $(g_1,\mathbf{h}_1)$.

6. Some examples

In order to see the need for some of the conditions imposed in the rank 2 theory, we discuss various possibilities in rank 1.

6.1. $r=1$ and two sections

Lets start with the situation of two sections $M_1\in\Gamma(S^+\otimes L^{\otimes q_1})$ and $M_2\in\Gamma(S^+\otimes L^{\otimes q_2})$, with the q_i odd. We take the equations to be given by (2.1) and (2.2) with $E_{11}=q_1$, $E_{21}=q_2$ and $E_{12}=E_{22}=0$. We set $D=E^{\mathrm{T}}$, so that $B_{11}=q_1^2$, $B_{12}=q_1q_2$ and $B_{22}=q_2^2$ and $\det B=0$. The virtual dimension of the moduli space is

$$d=d_0+d_1(L^{\otimes q_1})+d_1(L^{\otimes q_2})=-\frac{\chi+2\tau}{2}+(q_1^2+q_2^2)c_1(L)^2.$$

(4.5), with $A_2=0$, is the appropriate inequality in the present situation,

$$\int_X \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}|F^+|^2+\sum_{i=1}^{2}|D||M_i|^2+\frac{1}{2}(|q_1||M_1|^2-|q_2||M_2|^2)^2\right.$$
$$\left.+\frac{1}{4}R\sum_{i=1}^{2}|M_i|^2\right)\leq 0.$$

Unfortunately, one sees directly that along the line $|q_1||M_1|^2 = |q_2||M_2|^2$ we cannot deduce any bounds. We, can do a little better and work with the equality,

$$\int_X \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}\sum_{i=1}^{2}|F^{i+}|^2 + \sum_{i=1}^{2}|D\,M_i|^2 + \frac{1}{2}(|q_1||M_1|^2 - |q_2||M_2|^2)^2 \right.$$
$$\left. +2|q_1q_2||\bar{M}_1M_2|^2 + \frac{1}{4}R\sum_{i=1}^{2}|M_i|^2\right) = 0$$

and now it becomes transparent that problems of non-compactness come from the region where $|q_1||M_1|^2 \approx |q_2||M_2|^2$ and $|\bar{M}_1\cdot M_2| \approx 0$ as the norms of both sections become large. Of course, one needs a more explicit understanding of a given set of solutions to know if such situations arise, which in turn means that we do not have a general compactness theorem available. However, one thing that we do learn from this example is that the success in establishing compactness of the moduli space in the rank 2 case is rather non-trivial.[1] Equations of this type arise in the context of the twisted version $N = 2$ supersymmetric $SU(2)$ gauge theory with $N_f = 2, 3$ massless fundamental hyper-multiplets [13] (though we disagree with the vanishing theorem presented there).

These equations have been studied in the mathematics literature [6], with $q_1 = q_2 = 1$. More generally the authors consider a rank 1 theory with N sections and all charges unity. Even though the moduli space is non-compact they show that there is a natural compactification. Unfortunately, the dimension of this moduli space can never be zero. Notice that, in the current setting, the SW equations are invariant under $M_i \to U_{ij}\,M_j$ with $U \in SU(N)$. This $SU(N)$ symmetry is a global 'flavour' symmetry and has nothing to do with the group of gauge transformations. This means that there is a non-trivial action of $SU(N)$ on the moduli space. However, by allowing for charges q_i such that $q_i \neq q_j$ when $i \neq j$, there is no such symmetry, and it appears that the arguments presented in [6] still go through.

6.2. $r = 1$ *and one section*

Another possible set of equations is to consider one section $M \in \Gamma(S^+ \otimes L^{\otimes q})$ with q odd. The dimension in this case is

$$d = d_0 + d_1(L^{\otimes q}) = -\frac{2\chi + 3\tau}{4} + q^2\,c_1(L)^2.$$

We may define basic classes to be $y = -c_1(L^2)$ which satisfy $q^2y^2 = 2\chi + 3\tau$. Denote the corresponding invariant by n_y. In the usual SW equations, one considers a line bundle L' and for each $x = -2c_1(L')$ which obeys $x^2 = 2\chi + 3\tau$ one associates an integer n_x which, under certain conditions, is a topological invariant. The total of the available topological

[1] The lack of compactness persists if one has equations with more sections than connections regardless of the rank.

invariants is obtained on running over all possible line bundles. When $L' = L^{\otimes q}$ the moduli spaces and the invariants agree, $n_x = n_y$.

What we have learnt is that the invariants that are available for a monopole with a higher charge are a subset of those of charge one. There are two cases. For manifolds with $2\chi + 3\tau \neq 0$, one may chose q large enough so that there are no basic classes at all. This means that in this situation one may be able to 'fine tune' so that, by an appropriate choice of q, only a small subset of basic classes will arise. For manifolds with $2\chi + 3\tau = 0$, q plays no role in the dimension formula. It would be nice to find a way to use this mismatch in the dependence on q to learn something about topology.

6.3. $r = 2$ *and one section*

For our last example, we will consider in this section is that of rank 2 but with just one section $M \in \Gamma(S^+ \otimes L_1^{\otimes q_1} \otimes L_2^{\otimes q_2})$. We have $E_{11} = q_1$, $E_{12} = q_2$ and $E_{21} = E_{22} = 0$, and again we take $D = E^{\mathrm{T}}$ then the only non-zero component of B is $B_{11} = q_1^2 + q_2^2$. Note that $q_2 F^{1+} - q_1 F^{2+} = 0$ or, put another way, $q_2 A_1 - q_1 A_2$ is a self-dual Abelian instanton. However, as discussed previously, by the perturbation of the equations there are no solutions to the equations at all for $b_2^+(X) > 0$. So, we learn that we should have $n > r$.

Putting together the various pieces from these examples, we see that in fact the interesting case comes precisely when the rank r is equal to the number of sections n.

7. Kähler manifolds

If X is Kähler one has decompositions $S^+ \otimes \mathbf{L}_i = (K_X^{1/2} \otimes \mathbf{L}_i) \oplus (K_X^{-1/2} \otimes \mathbf{L}_i)$ where, as before, neither $K_X^{\pm 1/2}$ nor $\mathbf{L}_i$ necessarily exist. Denote the components of M_i in $K_X^{1/2} \otimes \mathbf{L}_i$ by α_i and those in $K_X^{-1/2} \otimes \mathbf{L}_i$ by $\sqrt{-1}\,\overline{\beta}_i$. The equations become

$$\begin{aligned} F_{\mathbf{B}_i}^{(2,0)} &= \alpha_i\, \beta_i \\ \omega \wedge F_{\mathbf{B}_i} &= \tfrac{1}{2}\omega^2(|\alpha_i|^2 - |\beta_i|^2) \\ \overline{\partial}_{\mathbf{A}_i}\alpha_i &= -i\overline{\partial}^*_{\mathbf{A}_i}\overline{\beta}_i \end{aligned} \tag{7.1}$$

The holomorphic description of this setting is as follows. First recall that the $\mathbf{B}_i$ are connections on the bundles $\mathcal{L}_i = L_1^{D_{i1}^{-1}} \otimes L_2^{D_{i2}^{-1}}$. The degree of a line bundle $\mathcal{L}$ is taken to be

$$\deg(\mathcal{L}) = \int_X c_1(\mathcal{L}) \wedge \omega. \tag{7.2}$$

Proposition 7.1. *Let (A_i, M_i) be a solution to the rank 2 SW equations with $M_i = (\alpha_i, \sqrt{-1}\,\overline{\beta}_i)$. For some i, if the degree of $\mathcal{L}_i$ is ≤ 0 then $\beta_i = 0$ and if the degree of $\mathcal{L}_i$ is ≥ 0 then $\alpha_i = 0$. Furthermore, the $\mathbf{B}_i$ induce a holomorphic structure on $\mathbf{L}_i$ and with respect to the induced holomorphic structures the sections α_i and β_i are holomorphic sections of $K_X^{1/2} \otimes \mathbf{L}_i$ and $K_X^{1/2} \otimes \mathbf{L}_i^{-1}$, respectively.*

Proof. The proof is analogous to the argument given by Witten for the rank 1 equations. The formula (4.3) with present notation and conditions is invariant under $A_i \to A_i$, $\alpha_i \to -\alpha_i$ and $\beta_i \to \beta_i$ performed for $i = 1$ and 2 simultaneously (this becomes rather more transparent on taking (4.4) into account). But this means that both $F^{(2,0)}_{\mathbf{B}_i} = \alpha_i\,\beta_i$ and $F^{(2,0)}_{\mathbf{B}_i} = -\alpha_i\,\beta_i$ are simultaneously zeros of (4.3) that is, if $(\mathbf{B}_i, \alpha_i, \beta_i)$ is a solution of the rank 2 equations then so too is $(\mathbf{B}_i, -\alpha_i, \beta_i)$. Consequently, the first equation in (7.1) becomes,

$$0 = F^{(2,0)}_{\mathbf{B}_i} = \alpha_i\,\beta_i, \tag{7.3}$$

which means that the line bundles $\mathcal{L}_i$ are holomorphic and that at least one of α_i and β_i is zero for each i. Notice that by linearity the line bundles L_i and $\mathbf{L}_i$ are also holomorphic. By the second equation in (7.1) we see that the degree of $\mathcal{L}_i$ and the vanishing of either α_i or β_i is as stated in the proposition. Lastly, we see that by the last equation in (7.1) that the sections are indeed holomorphic. □

To complete the holomorphic description of the moduli space of solutions, we interpret the second equation of (7.1) as a moment map for the group of gauge transformations. On the space of connections A_i introduce the symplectic form

$$\Omega(\delta_1 A, \delta_2 A) = \sum_{i,j} \int_X G^{ij}\, \omega \wedge \delta_1 A_i \wedge \delta_2 A_j. \tag{7.4}$$

Suppose that we are in the situation where both of the $\beta_i = 0$. On the space of sections (α_1, α_2) of $K_X^{1/2} \otimes (\mathbf{L}_1 \oplus \mathbf{L}_2)$ there is a symplectic structure

$$\Omega(\delta_1\alpha, \delta_2\alpha) = -\sqrt{-1}\sum_i \int_X \frac{\omega^2}{2}\,(\delta_1\overline{\alpha}_i\,\delta_2\alpha_i - \delta_2\overline{\alpha}_i\,\delta_1\alpha_i) \tag{7.5}$$

The space of connections and sections (A_i, α_i) can be interpreted as a symplectic manifold with symplectic form given by (7.4) and (7.5). Set,

$$\mu_i\,\omega^2 = E^{\mathrm{T}}_{ij}\,\omega(F_{\mathbf{B}_j} + \omega\,\overline{\alpha}_j\alpha_j) = \omega(G^{ij} F_{A_j} + E^{\mathrm{T}}_{ij}\,\overline{\alpha}_j\alpha_j\,\omega) \tag{7.6}$$

which is the moment map for the $U(1) \times U(1)$ gauge transformations. Morally, therefore, the space of solutions is the space of holomorphic sections modulo the induced action of the group of complex gauge transformations $\mathcal{G}_1^{\mathbb{C}} \times \mathcal{G}_2^{\mathbb{C}}$. Hence, in the case that $\deg(\mathcal{L}_i) < 0$ for $i = 1$ and 2 one expects that the moduli space of solutions is made up of two pairs, $(\mathbf{L}_i, \alpha_i)$, of a line bundle with a given hermitian structure and a non-zero holomorphic section of $K^{1/2} \otimes \mathbf{L}_i$ defined up to constant scaling.

Remark 7.2. When the degree of any of the bundles $\mathcal{L}_i$ is positive it is the associated section β_i which is non-zero. In this case, there is also a symplectic form analogous to that for the α_i available and the expectations are the same with $K^{1/2} \otimes \mathbf{L}_i$ replaced by $K^{1/2} \otimes \mathbf{L}_i^{-1}$.

While it is quite encouraging that one of the rank 2 equations is indeed a moment map for the gauge symmetry the question of what the right notion of stability is, in this context, is still open. In the rank 1 case, one can prove that indeed dividing through by the complexified gauge group is equivalent to setting the moment map to zero and dividing out by the usual gauge transformations. This is the content of Lemma 7.2.4 of [12]. So, in the case of the rank 1 SW equations, it is enough to have a holomorphic section of a holomorphic line bundle to solve both of the remaining SW equations. The proof of this statement, given in [12], uses a highly non-trivial result in analysis due to Kazdan and Warner [10]. Infact, the use of the Kazdan–Warner result in this context is origially due to Bradlow [3]. To give an analogous proof for the rank 2 equations would require a solution to a system of Kazdan–Warner type equations. Unfortunately, we do not know of a solution to such a system.

Since that result does not easily generalize we provide a weaker form of the statement available for the rank 1 equations which does generalize to the rank 2 setting. The alternative does not rely on the work of Kazdan–Warner for rank 1 which means that we are free to use the Kazdan–Warner theorem in rank 2.

The idea is to show that there are solutions to the SW equations for a metric in the conformal class of the Kähler metric. This only requires usual Hodge theory.

Proposition 7.3. *Let (ω, X) be a Kähler manifold and $(e^{2\rho}\omega, X)$ be X equipped with a metric conformal to the Kähler metric, with $\rho : X \to \mathbb{R}$. Suppose that the degree of a holomorphic line bundle $L^{\otimes 2}$ is negative and that B_0 is a hermitian holomorphic connection on $L^{\otimes 2}$. Suppose, also that α is a non-zero holomorphic section of $K^{1/2} \otimes L$. Then, for a particular conformal factor ρ (up to scalars) there exists another hermitian structure h on $L^{\otimes 2}$ such that for the connection B, which is hermitian with respect to $h = (\exp\rho) \cdot h_0$ and which defines the same holomorphic structure on $L^{\otimes 2}$ as B_0, that*

$$\omega \wedge F_B = e^{-\rho}\, |\alpha|_h^2\, \omega^2 = |\alpha|^2\, \omega^2.$$

Proof. The change in hermitian structure relates the curvatures by

$$F_B = F_{B_0} - i\bar{\partial}\partial\rho$$

so the equation that needs to be solved is

$$\omega \wedge F_{B_0} - i\omega \wedge \bar{\partial}\partial\rho = |\alpha|^2\, \omega \wedge \omega.$$

However,

$$i\omega \wedge \bar{\partial}\partial\rho = \Delta\rho\, \omega \wedge \omega$$

so that we want a solution to

$$\Delta\rho + |\alpha|^2 + C = 0,$$

where $C \wedge \omega^2 = -\omega \wedge F_{B_0}$. This last equation has a unique solution up to the addition of a constant. $\square$

Remark 7.4. Since we only make use of Hodge theory the proposition could equally well have been stated with *f* any positive semi-definite function replacing the square of the norm of the holomorphic section $|\alpha|^2$.

Proposition 7.5. *Suppose that the moduli space of SW equations on* (ω, X) *is zero dimensional. Then, given a holomorphic section one obtains a solution to the rank 1 SW equations on* (ω, X).

Proof. In the present setting the moduli space is a set of points and by the compactness of the space it must be a finite set, let them be denoted by p_a. The idea of the proof is that we can obtain each of these points as solutions to the SW equations on $(e^{2\rho_a}\omega, X)$, respectively. By Proposition 2.3, the section that solves the SW equations on $(e^{2\rho}\omega, X)$ can as well be taken to be the holomorphic section on (ω, X). By Proposition 7.3 given a holomorphic section of $K^{1/2} \otimes L$ we obtain a connection B on the holomorphic bundle $L^{\otimes 2}$ such that $\omega \wedge F_B = e^{-\rho}\, |\alpha|_h^2$ with ρ determined by the section. By Proposition 2.3, we have thus a solution to the SW equations on $(e^{\rho}\omega, X)$. By Proposition 5.5, the solution space (of the perturbed equations) is independent of the metric and so since we have a solution to the equations on $(e^{\rho}\omega, X)$ this must be continuously connected to a solution on (ω, X) in the space of connections and sections. □

Remark 7.6. This result is much weaker than Lemma 7.2.4 in [12]. Running through all possible holomorphic sections on (ω, X) we get a list of solutions on various Riemannian manifolds all conformally equivalent to the Kähler manifold (ω, X). All of these points must be points in the moduli space, since they solve the SW equations on the appropriate Riemannian manifold. However, we have no way of knowing if they are distinct. It could happen that one holomorphic section α_1 gives us a solution on $(e^{\rho_1}\omega, X)$ and another holomorphic section α_2 yields a solution on $(e^{\rho_2}\omega, X)$ and these points are continuously connected in the parametrized space of connections and sections. Of course, the lemma just cited tells us that this does not happen in the rank1 case.

We need a version of Lemma 7.2.4 of [12].

Proposition 7.7. *Suppose that the degree of a line bundle* $\mathcal{L}$ *is negative and that* B *is a hermitian holomorphic connection on* $\mathcal{L}$. *Let f be any positive semi-definite function. Then, there exists another hermitian structure* h' *on* $\mathcal{L}$ *such that for the connection* B' *which is hermitian with respect to* $h' = (\exp\lambda)\cdot h$ *and which defines the same holomorphic structure on* $\mathcal{L}$ *as* B *and, furthermore,*

$$F_{B'}^{(1,1)} = (\exp\lambda)\cdot f\cdot\omega$$

Proof. The curvatures are related by

$$F_{B'} = F_B - i\bar{\partial}\partial\lambda$$

so the equation that needs to be solved is

$$\omega \wedge F_B - i\omega \wedge \bar{\partial}\partial\lambda = \exp\lambda \cdot f \cdot \omega \wedge \omega.$$

However,

$$i\omega \wedge \bar{\partial}\partial\lambda = \Delta\lambda\, \omega \wedge \omega$$

and the equation that needs to be solved is

$$\Delta\lambda + \exp\lambda \cdot f + C = 0, \tag{7.7}$$

where $C\,\omega \wedge \omega = -F_B \wedge \omega$ and C is negative since the degree of $\mathcal{L}$ is. This equation is shown to have a unique solution for $\lambda : X \to \mathbb{R}$ in [10] as quoted in [12]. □

We can now show that there are solutions to the rank 2 equations.

Proposition 7.8. *Let (ω, X) be a Kähler manifold and $(e^{2\rho}\omega, X)$ be X equipped with a metric conformal to the Kähler metric, with $\rho : X \to \mathbb{R}$. Let $\mathcal{L}_i$ be two holomorphic line bundles on X and suppose that* $\deg(\mathcal{L}_i) < 0$ *for $i = 1, 2$. Let $\mathbf{B}_i^0$ be Hermitian holomorphic connections on the $\mathcal{L}_i$ and that α_i are non-zero holomorphic sections of $K_X^{1/2} \otimes \mathbf{L}_i$. For a particular conformal factor ρ and particular hermitian structures h_i on the same holomorphic bundles $\mathcal{L}_i$ there are hermitian connections $\mathbf{B}_i$ for which*

$$\omega \wedge F_{\mathbf{B}_i} = e^{-\rho}\, |\alpha_i|^2_{h_i}\, \omega^2$$

Proof. The equations, generalizing those in Proposition 7.3, that have to be solved are

$$\sum_j D_{ij}^{-1}(F_{A_j^0} - \Delta\lambda_j\, \omega) \wedge \omega = \exp\left(-\rho + \sum_j E_{ij}\lambda_j\right) \cdot |\alpha_i|^2\, \omega^2$$

or

$$\sum_j H_{ij}\, \Delta\mu_j + \exp(-\rho + \mu_i) \cdot |\alpha_i|^2 + C_i = 0, \tag{7.8}$$

with $G = E^{\mathrm{T}} \cdot H \cdot E$, $\mu_i = \sum_j E_{ij}\lambda_j$ and $C_i \wedge \omega^2 = -\sum_j D_{ij}^{-1} F_{A_j^0} \wedge \omega$. Since H is a positive definite matrix not both of H_{11} and H_{22} can be zero. Suppose, it is H_{11} that is not zero (otherwise repeat the following with the obvious exchanges). Set $\rho = \mu_1$ then

$$\mu_1 = -\frac{1}{H_{11}}\left(H_{12}\,\mu_2 + \frac{1}{\Delta}\left(|\alpha_1|^2 + C_1\right)\right)$$

solves the $i=1$ part of (7.8). Suppose that $H_{12} \neq -H_{11}$, then the $i=2$ part of (7.8) agrees with (7.7) with the following identifications:

$$\lambda = \left(1 + \tfrac{H_{12}}{H_{11}}\right) \mu_2,$$
$$C = \frac{H_{11}+H_{12}}{\det H} \left(C_2 - \frac{H_{21}}{H_{11}} \left(|\alpha_1|^2 + C_1 \right) \right),$$
$$f = \frac{H_{11}+H_{12}}{\det H} |\alpha_2|^2 \, e^{(1/H_{11})(1/\Delta)(|\alpha_1|^2 + C_1)}$$

If, on the other hand, $H_{12} = -H_{11}$ then the equation to solve is simply

$$\frac{\det H}{H_{11}} \Delta \mu_2 + g = 0,$$

where g is independent of μ_2, and, by Hodge theory, this has a solution. □

Remark 7.9. It is not clear why the choices $H_{12} = -H_{11}$ or $H_{12} = -H_{22}$ have such a privileged position. On the other hand if H is diagonal then one is dealing with two copies of the rank 1 SW equations.

When one bundle, say $\mathcal{L}_1$, is holomorphic and the other, $\mathcal{L}_2$, is anti-holomorphic the same arguments go through:

Proposition 7.10. *Let (ω, X) be a Kähler manifold and $(e^{2\rho}\omega, X)$ be X equipped with a metric conformal to the Kähler metric, with $\rho : X \to \mathbb{R}$. Let $\mathcal{L}_i$ be a holomorphic line bundle and $\mathcal{L}_j$ be an anti-holomorphic line bundle on X and suppose that* $\deg(\mathcal{L}_i) < 0$, *and* $\deg(\mathcal{L}_j) > 0$. *Let $\mathbf{B}_i^0$ and $-\mathbf{B}_j^0$ be Hermitian holomorphic connections on the lines $\mathcal{L}_i$ and $\mathcal{L}_j^{-1}$, respectively. α_i is a non-zero holomorphic section of $K_X^{1/2} \otimes \mathbf{L}_i$ and β_j is a non-zero holomorphic section of $K_X^{1/2} \otimes \mathbf{L}_j^{-1}$. For a particular conformal factor ρ and particular hermitian structures h_i and h_j on the same holomorphic bundles $\mathcal{L}_i$ and $\mathcal{L}_j$ there are hermitian connections $\mathbf{B}_i$ and $-\mathbf{B}_j$ for which*

$$\omega \wedge F_{\mathbf{B}_i} = e^{-\rho}\, |\alpha_i|^2_{h_i}\, \omega^2, \quad \omega \wedge F_{-\mathbf{B}_j} = e^{-\rho}\, |\beta_j|^2_{h_j}\, \omega^2.$$

What we have seen is that given a pair of holomorphic sections to $K^{1/2} \otimes \mathbf{L}_i$ on the Kähler manifold (ω, X) we are guaranteed a solution to the rank 2 SW equations on $(e^{\rho}\omega, X)$.

Now we come to perturbations. There are two types of perturbation adopted in the literature. The first, introduced by Witten, is to take $h \in \mathrm{H}^{(2,0)}(X) \oplus \mathrm{H}^{(0,2)}(X)$ which is geared to Kähler manifolds with $b_2^+(X) > 1$. The second option is that of Taubes [16] which is to set $h = r\omega$. Such a perturbation is available in the more general setting of almost Kähler manifolds, i.e. on symplectic manifolds with a compatible almost complex structure. We adopt Witten's perturbation.

We have the following:

Proposition 7.11. *Let X be a minimal surface of general type. Then, for any Kähler metric the basic classes of the rank 2 SW equations are a subset of the Cartesian product of the allowed rank 1 SW classes, i.e. subsets of the four classes* $(\pm K_X, \pm K_X)$. *If* $\exists\, L_i$ *such that* $\mathbf{L}_i = \pm K_X$ *then there are non-zero basic classes.*

Proof. We follow Witten's argument, footnote 11 on page of [17]. The perturbed equations require that the canonical bundle can be expressed as $K_X = \mathcal{O}(\Sigma_1) \otimes \mathcal{O}(\Sigma_2)$ and $K_X = \mathcal{O}(\Sigma_3) \otimes \mathcal{O}(\Sigma_4)$ however, Lemma 4 of Kodaira [14] tells us that if the Σ_i are non-zero effective divisors then $\Sigma_1 \cdot \Sigma_2 > 0$ and $\Sigma_3 \cdot \Sigma_4 > 0$. Denote the divisors of L_i by $[D_i]$ and $\bar{\Sigma} = (\Sigma_1 - \Sigma_2, \Sigma_3 - \Sigma_4)^{\mathrm{T}}$. Then, we have $\bar{\Sigma} = 2E \cdot [D]$ and

$$\bar{\Sigma}^{\mathrm{T}} \cdot \bar{\Sigma} = 4C_{ij}[D_i] \cdot [D_i].$$

The dimension formula gives us

$$4d = \bar{\Sigma}^{\mathrm{T}} \cdot \bar{\Sigma} - 2K_X^2,$$

however, $2K_X^2 = (\Sigma_1 + \Sigma_2)^2 + (\Sigma_3 + \Sigma_4)^2$ so that we have

$$d = -\Sigma_1 \cdot \Sigma_2 - \Sigma_3 \cdot \Sigma_4.$$

If the Σ_i are all non-zero then the lemma quoted above implies that the dimension is negative and so we have an empty moduli space. The same lemma tells us that the dimension cannot be greater than zero. The zero dimensional (and non-empty) moduli space requires that $\Sigma_1 \cdot \Sigma_2 = 0$ and $\Sigma_3 \cdot \Sigma_4 = 0$. This gives the four possibilities stated in the proposition.

Since there is precisely one section associated with each choice we have, by Proposition 7.8, that there is indeed a solution to the rank 2 equations if $\exists\, L_i$ such that $\mathbf{L}_i = \pm K_X$. $\square$

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